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The structure of WTC expansions and applications

Satyanad Kichenassamy and Gopala Krishna Srinivasan

University of Minnesota, 127 Vincent Hall, School of Mathematics, 206 Church Street SE, Minneapolis, MN 55455-0487, USA

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Abstract. We construct generalized Painlevé expansions with logarithmic terms for a general class of ('non-integrable') scalar equations, and describe their structure in detail. These expansions were introduced without logarithms by Weiss–Tabor–Carnevale (WTC). The construction of the formal solutions is shown to involve semi-invariants of binary forms, and tools from invariant theory are applied to the determination of the type of logarithmic terms that are required for the most general singular series. The structure of the series depends strongly on whether 1 is or is not a resonance. The convergence of these series is obtained as a consequence of the general results of Littman and Kichenassamy. The results are illustrated on a family of fifth-order models for water-waves, and other examples. We also give necessary and sufficient conditions for -1 to be a resonance.

1. Introduction

1.1. Background

In 1983, seeking a generalization of the Painlevé (or Painlevé–Kowalewski) test for integrability by inverse scattering, Weiss, Tabor and Carnevale (WTC) [21] showed that Burgers' equation, the Korteweg–de Vries equations and a few others possess formal solutions of the form

$$\phi(x,t)^{\nu}\sum_{j\geqslant 0}u_j(x,t)\phi(x,t)^j$$

in which the number of arbitrary coefficients is equal to the order of the equation minus one, and ν is negative. The values of j such that u_j is arbitrary are called *resonances*, and they are the roots of a polynomial which can be computed from the equation. It was rapidly noticed that the construction of this series is greatly simplified if one lets $\phi = t - \psi(x)$ (reduced ansatz, Kruskal), which essentially means that one may take ϕ as a new time variable.. The original formulation is sometimes more instructive, since one can in important cases derive a Bäcklund transformation and a Lax pair from it; ϕ is then related to the eigenfunction of the associated eigenvalue problem. The existence of such expansions has been proved for a large number of equations integrable by inverse scattering, suggesting that their existence is the basis of a test for integrability. Some equations do however have a commutator representation and admit solutions with more complicated movable singularities: the Harry Dym equation requires fractional powers of ϕ (see [7] for a possible explanation), while the Chazy equation, a reduction of the self-dual Yang–Mills equations, has solutions with a movable natural boundary [1]. The status of the WTC test is described in the surveys [16, 20, 17, 22, 8]; extensive references are contained in [16, 2, 8].

On the other hand, a large number of equations with polynomial nonlinearities have formal expansions of the form

$$\phi(x,t)^{\nu} \sum_{j \geqslant k \geqslant 0} u_{j,k}(x,t)\phi(x,t)^{j}[\ln \phi(x,t)]^{k}$$

and the previous series corresponds to the vanishing of the coefficients of the logarithmic terms. It seems that the presence of logarithms implies that the solutions in question have singularities which cluster in a self-similar fashion, and this is sometimes viewed as a possible symptom for non-integrable behaviour (see, for example, Levine and Tabor [17] and their references). It therefore becomes important to understand the structure of the series in more detail, since this seems to give some indication of the nature of 'non-integrable behaviour.' Truncations of such series sometimes give exact solutions in 'non-integrable' cases; see, for example, [18, 19]. Also, the consideration of logarithmic series sheds some light on the mechanism of singularity formation in semi-linear evolution equations (see [14]), integrable or not, and provides new paradigms.

In terms of the reduced ansatz, and taking ϕ as a time variable, these more general series will be written

$$t^{\nu} \sum_{j \geqslant k \geqslant 0} u_{j,k}(x)t^{j}(\ln t)^{k}.$$

They will be referred to in the rest of the paper as WTC expansions, and the reduced ansatz will always be used from now on.

1.2. *Issues and results*

(a) The convergence of WTC expansions with or without logarithms was proved in a quite general setting in [14]; the assumption is that the first term of the expansion can be found, and the conclusion is that there is an integer l and at least one series

$$t^{\nu} \sum_{j_0,\dots,j_l \geqslant 0} u_{j_0,\dots,j_l}(x)t^{j_0}[t\ln t]^{j_1}\cdots[t(\ln t)^{l}]^{j_l} \tag{1}$$

which converges for small $|t|$ and solves the equation. A constructive procedure for estimating l and for computing the coefficients follows from the proof. The convergence follows from the existence of analytic solutions for a 'generalized Fuchsian equation'. The procedure is recalled, with a slight improvement, and applied to the equations of this paper in section 4. The argument applies in any number of space dimensions, to equations as well as systems. Note that another method for proving convergence in the case of the Korteweg–de Vries equation was announced in [13]. It is based on a reduction to an iteration adapted to this equation. It seems that another reduction, adapted to Burgers' equation, is possible. Several results for ODEs have been known for some time (see especially [3], which deals specifically with the Painlevé test).

(b) The number l ('number of logarithms') in (1) was estimated rather crudely in [14]. For scalar equations of high order, it can be wide of the mark since it rests on the preliminary reduction to a large first-order system, convenient for the convergence proof. We give a more realistic estimate for single equations, which is optimal in several cases. Thus, $l = 1$ suffices if all resonances are simple and 1 is not a resonance. We also briefly show that the logarithmic series can also sometimes be viewed as a series in t and $t^{m}\ln t$, where m can be estimated explicitly; here, the spacing of the resonances and the form of the nonlinear terms must be taken into account. Such a formulation comes up in connecting the presence of logarithms with the existence of self-similar clusters of singularities; see, for example, Levine and Tabor [17].

(c) It is well known that -1 is often, but not always, a resonance. Its occurrence can be formally explained using the arbitrariness of the singularity surface. -1 is not a resonance in the case of the Cauchy problem (WTC expansion with $\nu = 0$, and no logarithms). Clarkson and Cosgrove [6] give a number of enlightening examples, and suggest that -1 is not a resonance if, upon substitution of the series into the equation, only terms involving u_0 occur in the most singular terms, and if setting their sum equal to zero produces a non-trivial equation for u_0. We show that this is correct by giving a necessary and sufficient condition for -1 to be a resonance (see section 2).

(d) We apply these results in section 5 to a class of fifth-order model equations which occur in water wave models and several other applications (see Kichenassamy and Olver [15] for many references). Only two sets of parameter values (apart from the known integrable cases) had previously been investigated from the point of view of the WTC method: (i) Jeffrey and Xu [12] considered the case when $\nu = -4$, which is somewhat exceptional, most parameter values leading to $\nu = -2$. They found that pure power expansions do not exist in general, by computing the compatibility condition at level 8. (ii) Conte *et al* [8] found one other case where four non-negative resonances occur. As we show, there are, for general parameter values, 18 cases where there are four positive resonances for one choice of u_0; for four of them only does the other choice of u_0 also lead to the maximum number of positive resonances (viz three) including the Sawada–Kotera, Kaup–Kuperschmidt and fifth-order KdV equations. None of the other cases leads to series which are entirely free of logarithms. There are nine further cases if we consider non-negative resonances. Values of l for these equations can however be determined for all, and the results are summarized in table 1. For some parameter values, the equation degenerates to third order, and can in some cases have series solutions with two arbitrary coefficients. These degenerate cases are also interesting in their having a second WTC series with $\nu = 1$, which is *not* of Cauchy–Kowalewska type. This example is similar to those of Clarkson–Cosgrove. A few other peculiarities are also noted.

(e) An important tool will be the analysis of the operator

$$M = t_0\partial/\partial t_0 + (t_1 + t_0)\partial/\partial t_1 + \cdots + (t_l + lt_{l-1})\partial/\partial t_l$$

acting on homogeneous polynomials in $(t_0, \ldots, t_l)$. Remarkably enough, the equation $Mu = 0$ expresses that u is a *semi-invariant* (also known as a *source of covariants*) in the sense of the invariant theory of binary forms (see [11], the introduction to which contains many modern references). The necessary material on invariant theory is included in the appendix of the present paper. The use of properties of M streamlines the construction of the WTC series. Note that the operator M also arises, in a somewhat different context, in the construction of normal forms near critical points with a nilpotent linear part [9, 10].

1.3. Organization of the paper

Section 2 contains a more technical description of the WTC algorithm (with logarithms) for scalar equations, and examines when -1 can be a resonance. It also contains some results which are used in section 3.

Section 3 gives general results on the form of WTC expansions with logarithms, and shows how their convergence follows from the results of [14], via a reduction to a Fuchsian system. This section also contains a reduction of general semi-linear systems to Fuchsian form, which complements the results of section 2.

Section 4 shows gives better estimates for the 'number of logarithms', based on properties of the operator M.

Section 5 applies the previous results to specific examples, which also illustrate possible pathologies.

The appendix proves the properties of M that are needed in section 4, and outlines the relation to invariant theory.

2. The WTC algorithm

The WTC algorithm seeks singular solutions with power growth, for PDE with polynomial-type nonlinearities. *The singularity is localized on a surface, near which the solutions* behave like a power of the distance to the surface. The leading behaviour is determined in such a way that the top-order derivatives balance some of the nonlinear terms.

For simplicity, we consider only scalar equations with polynomial dependence on the unknown and its derivatives; it is not difficult to extend our considerations to rational nonlinearities. The equation reads

$$F[u] := F(t, x_1, \dots, x_n, u, \partial_t u, \partial_{x_1} u, \dots) = 0. \tag{2}$$

After a change of variables, we assume that the singularity occurs at $t = 0$. Let m be the order of the equation, which will also be assumed to be the order of the highest time derivative. This means that the singularity surface is non-characteristic. All considerations are local, near $(x, t) = (0, 0)$.

The solution will be of the form $u = u_0(x)t^\nu(1 + o(1))$ as t tends to zero, with $u_0 \neq 0$. More precisely, the original WTC test requires the existence of solutions of the form

$$u(x, t) = \sum_{j \geqslant 0} u_j(x) t^{\nu+j} \tag{3}$$

while the weak Painlevé test requires

$$u(x, t) = \sum_{j \geqslant 0} u_j(x) t^{\nu+j/q} \tag{4}$$

for some integer q; one usually also requires ν to be a fraction $-p/q$, with $\gcd(p, q) = 1$. On the other hand, 'non-integrable' cases usually lead to the more general expansion

$$u(x, t) = \sum_{j_0, \dots, j_l \geqslant 0} u_{j_0, \dots, j_l}(x) t^{\nu+j_0} (t \ln t)^{j_1} \cdots (t(\ln t)^l)^{j_l}. \tag{5}$$

We will see that the latter is indeed the most general singular expansion in many cases. Equation (4) can of course be subsumed in principle under (3) by taking $t^{1/q}$ as new time variable. For the same reason, we do not consider expansions (5) involving fractional powers of t. Like most authors, we also exclude logarithms in the leading terms.

2.1. Leading term

Since t plays a special role, it is appropriate to distinguish space and time variables; we introduce some notation which reflects this concern. Let $\partial_x^I = \partial_{x^1}^{i_1} \cdots \partial_{x^n}^{i_n}$ denote the most general space derivative; $I = (i_1, \dots, i_n)$ is a multi-index. The most general nonlinear combination of u and its derivatives is

$$u^a := \prod_{j,I} (\partial_t^j \partial_x^I u)^{a_{j,I}}. \tag{6}$$

Here, $a_I = (a_{1,I}, \dots, a_{m,I})$ is again a multi-index; note that pure time derivatives correspond to $i_1 = \dots = i_n = 0$. We define

$$g(a_I) = \sum_j a_{j,I} \qquad p(a_I) = \sum_j j a_{j,I}. \tag{7}$$

They will be called respectively the *degree* and *weight* of a_I. We also let $g(a) = \sum_I g(a_I)$, $p(a) = \sum_I p(a_I)$, and $|I| = i_1 + \cdots + i_n$. It is helpful to introduce a special notation for those monomials in F which do not contain space derivatives:

$$u^A = u^{A_0}(u_t^{A_1}) \cdots (\partial_t^m u)^{A_m}.$$

Since $A = (A_0, \ldots, A_m)$ is itself a multi-index, one can as before define its degree and weight. They correspond to those monomials (6) for which $I = (0, \ldots, 0)$.

We may now write the equation in the form

$$F[u] := \sum_{a=(a_I)} f_a(x,t) u^a = 0 \tag{8}$$

where

$$f_a(x,t) = \sum_{b \geqslant 0} f_{ab}(x) t^{b+\mu(a)}$$

and $f_{a0} \neq 0$. To minimize technicalities, we will assume that F is polynomial in u and its derivatives, so that the sum in (8) is finite. It is however possible to allow more general nonlinearities.

If $u = u_0(x)t^\nu + \text{h.o.t.}$, where h.o.t. refers to higher-order terms in t, we have

$$\partial_t^j \partial_x^I u = \nu(\nu-1)\cdots(\nu-j+1)(\partial_x^I u_0) t^{\nu-j} + \text{h.o.t.}$$

Therefore

$$\prod_j (\partial_t^j \partial_x^I u)^{a_{j,I}} = c(\nu, a_I) t^{\nu g(a_I) - p(a_I)} \prod_j (\partial_x^I u_0)^{a_{j,I}} + \text{h.o.t.}$$

where

$$c(\nu, a_I) = \prod_j [\nu(\nu-1)\cdots(\nu-j+1)]^{a_{j,I}}.$$

It follows that

$$f_a(x,t) \prod_{j,I} (\partial_t^j \partial_x^I u)^{a_I} = t^{\mu(a)+\nu g(a)-p(a)} f_{a0}(x) c(\nu, a) \prod_{j,I} (\partial_x^I u_0)^{a_{j,I}} + \text{h.o.t.} \tag{9}$$

where

$$c(\nu, a) := \prod_I c(\nu, a_I).$$

2.2. General strategy

We are now in a position to outline the line of attack.

We are interested in constructing solutions of the form $u = u_0 t^\nu + \text{h.o.t.}$, representing a *balance of the top-order time derivatives and some nonlinear term.*

We will first determine ν such that one may choose u_0 to satisfy the equation at lowest order. To simplify the equation for u_0, we assume that the most singular terms one obtains upon substitution of (3) or (4) into the equation *never contain any space derivatives*, and that the top-order time derivatives enter only into the most singular terms. This enables us to write

$$F[u_0(x)t^\nu + \text{h.o.t.}] = t^\rho (P(u_0) + \text{h.o.t.})$$

where

$$\rho = \mathrm{Min}_A\{\nu g(A) - p(A) + \mu(A)\} \tag{10}$$

and

$$P(u_0) := \sum_{\nu g(A)-p(A)+\mu(A)=\rho} f_{A0}(x)c(\nu, A)u_0^{g(A)}. \tag{11}$$

Thanks to our assumption, since no spatial derivatives enter at leading order, the leading term is determined by an *algebraic* equation ($P(u_0) = 0$), instead of a *differential* equation.

Once u_0 has been chosen among the roots of P, instead of constructing directly a recurrence relation for the higher-order terms in the expansion of the putative solution, it will be more efficient to show that there is a new unknown w, related to u by a formula of the form

$$t^{-\nu}u = u_0 + \sum_{q\leqslant k_0} h_q(x)t(\ln t)^q + tw(x,t) \tag{12}$$

which solves a *Fuchsian equation*

$$Q(x, t\partial_t)w = \sum_{q\leqslant 2k_0} t(\ln t)^q G_q[w]$$

where Q is a polynomial in its second argument, and the integer k_0 will be determined later. To this end, we will further require that the second most singular terms also do not involve space derivatives; this second assumption is not essential but simplifies the procedure; the most general statement will be given elsewhere.

Such an equation is said to be *Fuchsian* because it reduces to an ODE with a regular singular point at $t = 0$, in the event that the G_q do not depend on derivatives of w in the variables x.

The inductive construction of a formal solution of this equation will then be straightforward, the polynomial Q being related to the 'resonance equation' as explained in subsection 2.3.

For the needs of the proof of convergence of these series, we will establish that

$$G_q = G_q(x, t, \ldots, t(\ln t)^{l_0}, \{D^j w\}_{j<m}, \{tD^j \partial_x^J w\}_{j+|J|\leqslant m,\, k<m}).$$

Note the extra factor t in the derivative terms, which will be important in section 3.

For the more detailed study of the structure of the formal solution in section 4, we also mention that the number l_0 of logarithmic terms in G_q is twice the multiplicity of 0 as a root of $r \mapsto Q(x, r)$ (or twice the multiplicity of 1 as a 'resonance', as defined in subsection 2.3).

Before turning to the execution of this programme, let us close these preliminaries with a definition.

Definition. We say that ν is an admissible *balance* if there is a non-zero u_0 which satisfies $P(u_0) = 0$. Solutions corresponding to the same value of u_0 are said to belong to the same *branch.*

Remarks. (1) The restriction that no derivative terms occur at lowest order ensures that not only the equation for u_0, but also the recursion relation for the higher-order coefficients, be algebraic rather than differential equations.

(2) The definition means that it is reasonable to hope for a solution of the form $u = u_0 t^\nu + \text{h.o.t.}$

(3) In many cases, one determines ν by requiring that the minimum in (10) be attained for two values of A, the corresponding monomials in F balancing each other.

(4) The case $P(u_0) \equiv 0$ is somewhat degenerate, but occurs quite frequently, e.g., if $\nu = 0$ and $\{t = 0\}$ is non-characteristic (Cauchy problem). Another example is studied in subsection 5.3.

2.3. *Resonances and reduction to a Fuchsian equation*

Let us fix u_0 among the roots of P. We assume that we are not in the case of the Cauchy problem, so that $\nu(\nu-1)\cdots(\nu-m+1)\neq 0$.

We prove that under fairly general circumstances, the substitution (12) leads to a Fuchsian equation for w. In fact, we will establish that

$$F\left[t^{\nu}\left(u_0+\sum_{q\leqslant k_0}h_q(x)t(\ln t)^q+tw(x,t)\right)\right]$$
$$=t^{\rho}\left[P(u_0)+t\left\{Q(x,D)w-\sum_{q\leqslant 2k_0}t(\ln t)^qG_q\right\}\right]$$

if the h_q and k_0 are chosen suitably. More precisely, we have the following theorem.

Theorem 1. (a) After performing the substitution (12), equation (2) is equivalent to an equation of the form

$$Q(x,t\partial_t)w=\varphi(x)+\sum_{q\leqslant l_0}t(\ln t)^q$$
$$\times G_q(t,t\ln t,\ldots,t(\ln t)^{l_0},x,w,\ldots,D^{m-1}w,\{tD^k\partial_x^Jw\}_{k+|J|\leqslant m,\,k<m}) \tag{13}$$

where $D=t\partial_t$, for a suitable integer l_0.

(b) One has the following explicit formula for Q:

$$Q(x,r)=\sum_{\nu g-p+\mu=\rho}c(\nu,A)f_{A0}$$
$$\times\left[A_0+\frac{\nu+r}{\nu}A_1+\cdots+\frac{(\nu+r)\cdots(\nu+r-j+1)}{\nu\cdots(\nu-j+1)}A_j\right]u_0^{g(A)-1}. \tag{14}$$

(c) If $Q(x,D)=D^sR(x,D)$ with $R(x,0)\neq 0$, one can choose k_0 and the functions h_q in such a way that $\varphi=0$. One can in fact take $l_0=2k_0=2s$.

In particular, if $Q(x,0)\neq 0$, no logarithms are required on the r.h.s.

(d) The question of existence of a formal solution of the form (3) then reduces to the solution of a recurrence relation of the form

$$Q(x,r)w_{r+1}=F_r[w_0,\ldots,w_r]$$

where the expressions F_r can be computed recursively and may involve spatial derivatives of their arguments.

Remark. As already mentioned, an equation such as (13) is called *Fuchsian*, because it reduces to an ODE with a regular singular point at the origin if no x-derivatives are present. This form will be convenient to prove the convergence of formal series solutions in section 3. It is for this purpose that we insist on the derivative terms in the r.h.s. to come only in the combination $tD^k\partial_x^Jw$, instead of $D^k\partial_x^Jw$. The presence of the logarithms is due to the fact that we also need the r.h.s. to vanish for $t=0$. This will be achieved only by choosing suitably the coefficients h_q.

Proof. *Step 1. First change of unknown.* Let $u = t^{\nu} v(x,t)$, and $D = t\partial_t$. We have, by induction on j,

$$\partial_t^j u = t^{\nu-j}(D+\nu)\cdots(D+\nu-j+1)v.$$

Therefore

$$u^a = \prod_{j,J}\left(t^{\nu-j}(D+\nu)\cdots(D+\nu-j+1)\partial_x^J v\right)^{a_{j,J}}$$
$$= t^{\nu g(a)-p(a)}\prod_{j,J}\left[(D+\nu)\cdots(D+\nu-j+1)\partial_x^J v\right]^{a_{j,J}}.$$

Substituting into the equation and setting $t=0$, one recovers the equation $P(u_0)=0$ for u_0.

Step 2. Introduction of logarithms and second change of unknown. Fixing u_0 among the roots of P, we now let

$$v = u_0 + \sum_{q\leqslant k_0} h_q(x)t_q + tw(x,t)$$

where $t_q = t(\ln t)^q$, and the h_q, as well as the integer k_0, will be determined below. We find

$$(D+\nu)\cdots(D+\nu-j+1)\partial_x^J v$$
$$= \nu(\nu-1)\cdots(\nu-j+1)\partial_x^J u_0$$
$$+t_0(D+\nu+1)\cdots(D+\nu-j+2)\left[\partial_x^J w + \sum_q \partial_x^J h_q(\ln t)^q\right].$$

Note that this expression can be thought of as a first-degree polynomial in $(t_0,\ldots,t_{k_0})$, with coefficients involving functions of x, and derivatives of w of the form $tD^k\partial_x^J w$.

Step 3. Substitution into (6). Let us now consider what happens upon substitution into each term of F. The result is a series in the t_q, where the most singular term is $t^{\rho}P(u_0)$.

In a nutshell, we need to substitute and divide the equation by $t^{\rho+1}$. The result will contain linear contributions in w, which generate the terms in $Q(x,D)w$ in theorem 1, terms in logarithms, containing only the h_q, and higher-order terms. We need to factor an extra power of t in these terms. The terms involving space derivatives of w will immediately have such a factor, because they only contribute, by assumption, terms in $t^{\rho+k}$, $k\geqslant 2$. As for the others, the desired factor arises from products of $tD^j w$ terms, or from products $t_q t_{q'}$. They therefore end up having the form $t\times t(\ln t)^{q+q'}\times\Phi(x,t,\{D^j w\})$. This yields the desired form of the equation.

More precisely, we have, from step 2,

$$u^a = t^{\nu g(a)-p(a)}\Phi_a(x,t,\{t_q\partial h_q\},\{tD^j w\}_{j\leqslant m},\{tD^j\partial^J w\})$$

where ∂ stands for all space derivatives.

We now substitute this result into (6), which produces an expression of the form $t^{\rho}P(u_0)+\mathrm{O}(t^{\rho+1}(\ln t)^{2k_0})$. We need to divide this by $t^{\rho+1}$, since the sum of the terms in t^{ρ} vanishes by the choice of u_0.

To clarify the form of the result of this operation, we consider each term $f_a u^a$ separately. Each such term contributes terms of degree $\nu g(a) - p(a) + \mu(a)$, or higher. We also know that $\nu g(a) - p(a) + \mu(a) \geqslant \rho$, and this sum equals ρ or $\rho + 1$ only for terms which do not contain spatial derivatives.

The terms such that $\nu g(a) - p(a) + \mu(a) \geqslant \rho + 2$ still have a factor of t left after division by $t^{\rho+1}$, and therefore already have the desired form. For the others, we will use the Taylor expansion of u^a up to second order to extract the contributions in t^ρ and $t^{\rho+1}$.

We therefore only need to consider two types of terms.

(1) Those monomials with $\nu g(a) - p(a) + \mu(a) = \rho$; they contribute

$$t^\rho \left[P(u_0) + t\left(Q(x, D)\left[w + \sum_q h_q (\ln t)^q \right] + \varphi_1(x) \right) \right.$$
$$\left. + t \sum_{q \leqslant 2k_0} t(\ln t)^q \Psi_{1q}(x, \{t_q\}, \{D^j w\}_{j \leqslant m}) \right].$$

By inspection, the operator Q is as given in (14). φ_1 is some function of x.

(2) Those monomials with $\nu g(a) - p(a) + \mu(a) = \rho + 1$; in that case, we find a contribution

$$t^{\rho+1} \left[\varphi_2(x) + t \sum_{q \leqslant 2k_0} t(\ln t)^q \Psi_{2q}(x, \{t_q\}, \{D^j w\}_{j \leqslant m}) \right].$$

The function φ_2 depends only on x.

Combining these equations, we reach the desired assertion.

Step 4. Choice of k_0 and (h_q). We now finish the proof by showing that one can choose k_0 and (h_q) to eliminate $\varphi(x)$ under the assumption of case (b) in theorem 1. We have to solve

$$D^s R(x, D) \sum_{q \leqslant k_0} h_q (\ln t)^q + \varphi = 0$$

where φ is independent of t. Therefore, we need

$$R(x, D) \sum_{q \leqslant k_0} h_q (\ln t)^q + \frac{(\ln t)^s}{s!} \varphi = 0.$$

It is easy to see that there is a solution if $R(x, 0) \neq 0$ and $k \geqslant s$, which contains s arbitrary constants.

The theorem is proved, with $l_0 = 2k_0 = 2s$, as announced. □

Remarks. (1) The equation $Q(x, r - 1) = 0$ is called the *resonance equation* and its roots *resonances*. Nothing prevents resonances from varying with x. However, usually, u_0 is constant, and so are the resonances. Note that the resonance equation is not $Q(x, r) = 0$ because the initial value for w in (13) is in fact the *second term* in the expansion of u, u_0 being the first. If r is a resonance, the condition $F_{r-1} = 0$ is called the compatibility condition (at level r). The resonance is said to be *compatible* if this compatibility condition holds identically.

(2) It follows from the proof that $k_0 = (l_0/2)$ equals the multiplicity of 1 as a resonance.

2.4. Is −1 a resonance?

We give a necessary and sufficient condition for $Q(r-1)$ to be equal to zero for $r=-1$. We first choose u_0 such that $P(u_0)=0$.

Theorem 2. Assume that $\nu \neq 0, 1, \ldots, m-1$. Then, $Q(-2)=0$ if and only if

$$\sum_{\nu g - p + \mu = \rho} c(\nu, A) f_{A0} \mu(A) u_0^{g(A)} = 0. \tag{15}$$

This holds in particular if $\mu(A)$ is independent of A, i.e. if t does not enter explicitly in the balancing terms.

Proof. We compute $Q(-2)$:

$$\begin{aligned}
u_0 Q(-2) &= \sum c(\nu, A) f_{A0} u_0^{g(A)} \left[A_0 + \frac{\nu-1}{\nu} A_1 + \frac{(\nu-1)(\nu-2)}{\nu(\nu-1)} A_2 + \cdots\right] \\
&= \sum c(\nu, A) f_{A0} u_0^{g(A)} \left[A_0 + \left(1 - \frac{1}{\nu}\right) A_1 + \left(1 - \frac{2}{\nu}\right) A_2 + \cdots\right] \\
&= \sum_{\nu g(A) - p(A) + \mu(A) = \rho} c(\nu, A) f_{A0} u_0^{g(A)} [g(A) - p(A)/\nu] \\
&= \sum c(\nu, A) f_{A0} u_0^{g(A)} [\rho - \mu(A)]/\nu \\
&= P(u_0)/\nu - \frac{1}{\nu} \sum c(\nu, A) f_{A0} \mu(A) u_0^{g(A)}
\end{aligned}$$

from which the result follows. □

3. Convergence results

We are interested in constructing convergent series solutions

$$w = \sum_{a_0, \ldots, a_l} w_{a_0, \ldots, a_l} t^{a_0} (t \ln t)^{a_1} \cdots (t [\ln t]^l)^{a_l} \tag{16}$$

of (13) for an appropriate value of l.

To this end, we note that if we view w as a function of $(x, t_0, \ldots, t_l)$, and let

$$N = \sum_{k=0}^{l} (t_k + k t_{k-1}) \partial / \partial t_k = t_0 \partial_{t_0} + (t_1 + t_0) \partial_{t_1} + \cdots \tag{17}$$

then w solves

$$Q(N) w = \sum_q t_q G_q [w, Nw, \ldots]. \tag{18}$$

Such an equation is called a *generalized Fuchsian equation.*

Indeed, by the chain rule, one has, for any function w, that

$$D[w(t, t \ln t, \ldots, t (\ln t)^{l_0})] = (Nw)(t, t \ln t, \ldots, t (\ln t)^{l_0}).$$

It therefore suffices to seek solutions of (18): let us seek u in the form

$$u = \sum_a u_a t^a := \sum_{a_0, \ldots, a_l} u_{a_0, \ldots, a_l}(x) t_0^{a_0} \cdots t_l^{a_l}.$$

To prove the existence and convergence of series solutions for the equations of theorem 1, we follow the general strategy of [14] with minor improvements, see [14] for omitted proofs. The result will be a solution of the desired form, with some large value of l. The following section will show how to reduce the value of l.

We begin by proving (subsection 3.1) that (18) can be replaced by a Fuchsian system of the form

$$(N + A)u = \sum_q t_q G_q.$$

We then show (subsection 3.2) that all such systems have holomorphic solutions provided that the eigenvalues of A have positive real parts, and then give a procedure (subsection 3.3) whereby one can increase the eigenvalues of A by going to another extended system. This requires that l be sufficiently large. Applying this method to the system derived from (11), we conclude the existence and convergence of a formal series solution to (13).

Finally, we give two general cases when this reduction is possible.

The first is a general reduction theorem for semi-linear *systems* such as are found in the theory of solitons, for instance (subsection 3.4). It provides a second, very general, proof of the existence of formal solutions, but is not convenient to determine the optimal value of l.

The second (subsection 3.5) is the case in which we are given the existence of an approximate solution to a very high order; we show that this information ensures that an infinite formal series exists, and converges; thus, the existence of a formal series is shown to imply its convergence. This will be useful in one of the examples, where we will be able to construct a formal solution in a case when the procedures of section 2 or subsection 3.4 do not apply directly.

3.1. Reduction to a Fuchsian system

In the present situation, we start from the Fuchsian equation (13) and introduce the new unknown

$$(w, \ldots, D^{m-1}w, \{tD^k \partial_x^J w\}_{k+|J|<m})$$

where m is the order of the equation.

We proceed to compute the action of D on each of the new unknowns, taking (13) into account.

Let $w_k = D^k w$ and $tD^k \partial_x^J w = w_{k,J}$. We have

$$Dw_k = w_{k+1} \tag{19}$$

for $k + 1 < m$. On the other hand, let $\partial^J = \partial_{j_1}\partial_{j_2}\cdots = \partial_{j_1}\partial_x^{J'}$, with $j_1 \leqslant j_2, \ldots$

If $k + 1 + |J| < m$, we write

$$Dw_{k,J} = w_{k,J} + w_{k+1,J}. \tag{20}$$

If $k + 1 + |J| = m$, we write

$$Dw_{k,J} = w_{k,J} + tD^{k+1}\partial_x^J w = w_{k,J} + t\partial_{j_1} w_{k+1,J'}. \tag{21}$$

For the last derivative, namely $D(D^{m-1}w)$, we will use equation (13).

We first note that any $tD^k\partial_x^J w$ with $k + |J| = m$ and $k < m$ can be expressed as a first-order spatial derivative of one of our unknowns. We then write Q as

$$Q(x, D) = D^m + Q_1 D^{m-1} + \cdots$$

and find

$$Dw_{m-1} + \sum_{j>0} Q_j w_{m-j} = \sum_q t_q G_q(x, t_0, \ldots, w, \{\partial_j w_{k,J}\}). \tag{22}$$

Equations (19)–(22) now form a Fuchsian system where A may depend on x. In practice, u_0 and the f_{A0} are constant, and so are the coefficients Q_j.

3.2. Convergence theorem

The next point is that a generalized Fuchsian system

$$(N + A)u = \sum_{j=0}^{l} t_j f_j(t_0, \ldots, t_l, x, u, \partial_x u) \tag{23}$$

in any number of space dimensions, and for any l, has exactly one solution analytic near $t = 0$, $x = 0$, provided that f is analytic, and all the eigenvalues of A have positive real parts (theorem 3 in [14, part II], or [14, part I] if $l = 0$). This theorem contains the Cauchy–Kowalewska theorem as a special case, since we may convert

$$u_t = F(t, x, u, \partial_x u)$$

to the Fuchsian form

$$tu_t = tF(t, x, u, \partial_x u).$$

However, it does not follow from the Cauchy–Kowalewska theorem, which would predict not one but infinitely many solutions depending on the initial data.

3.3. Increasing the eigenvalues of A

Another general fact (see [14]) is that if we start from a Fuchsian system with arbitrary constant A, of the form

$$(N + A)u = \sum_{q \leqslant k_0} t_q f_q(t_0, \ldots, t_l, x, u, \partial_x u) \tag{24}$$

one can, if l is large enough, produce another system of the same form, the solutions of which generate solutions of (23), but in which the eigenvalues of A *have been raised by one*. Iterating the procedure, we may reduce ourselves to the situation of step 1 in finitely many steps. We will write ∂ for ∂_x.

More precisely, one seeks u in the form

$$u = u_0(x) + t \cdot v(x, t) = u_0 + t_0 v_0 + \cdots + t_l v_l. \tag{25}$$

We must choose u_0 in the null-space of A. Note that the new unknown v has $(l + 1)$ times as many components as u. Substituting, we find that

$$(N + A) \sum_{j=0}^{l} t_j v_j = \sum_{j=0}^{l} t_j \{(N + A)v_j + v_j + (j + 1)v_{j+1}\}$$

where v_{l+1} is taken to be zero, and

$$t_q f_q(t, x, u_0 + t \cdot v, \partial(u_0 + t \cdot v)) = t_0 \left(f_q(0, x, u_0, \partial u_0) + \sum_{j=0}^{l} t_j g_{qj}(t, x, v, \partial v) \right).$$

We are therefore led to require that v solve the system

$$(N + A + 1)v_j + (j+1)v_{j+1} = \delta_{jq} f_q(0, x, u_0, \partial u_0) + \sum_q t_q g_{qj}(t, x, v, \partial v) \tag{26}$$

(where δ_{j0} is the Kronecker symbol.) Clearly, any solution of (26) generates a solution of (24), via (25).

We now need to absorb $\delta_{jq} f_q(0, x, u_0, \partial u_0)$ into v. *This is where the value of l matters.* In fact, we need to be able to solve the system for the initial value of v, that is

$$(A+1)v_j + (j+1)v_{j+1} = \varphi(x) := \sum_{q \leqslant k_0} \delta_{jq} f_q(0, x, u_0, \partial u_0).$$

We may decompose v along two complementary subspaces, where $A+1$ is invertible and nilpotent, respectively. The invertible part is solved immediately ($v_0 = (A+1)^{-1}\varphi(x)$ and all the other $v_j = 0$). We therefore assume $(A+1)$ is nilpotent. We may then take $v_0 = 0$ and solve for the other v_j recursively. Since f_q vanishes for $q > k_0$, we have

$$v_j = \frac{[-(A+1)]^{j-k_0-1} v_{k_0+1}}{j(j-1)\cdots(k_0+2)}$$

for $j > k_0 + 1$, and the last equation reduces to

$$(A+1)^{l-k_0-l} v_l = 0$$

which holds for l large enough if A is nilpotent. Thus, if l has been chosen large enough at the outset, one may raise all the eigenvalues of A by 1 by considering (26) instead of (24). Since A has at most finitely many non-negative integer eigenvalues, we may reduce ourselves to the situation of step 1 in finitely many steps.

3.4. Semi-linear systems

We now show that rather general semi-linear systems can be cast in the form (24), as soon as the first term of a WTC-like expansion has been found. This proves at the same time the existence and the convergence of WTC expansions for such systems.

A crude estimate on the number l of logarithmic variables can be determined by following step 2. We include the details for the convenience of the reader, since they are not very lengthy.

The system has the form

$$u_t = \sum_{j=1}^{n} a^j \partial_j u + b(u) \tag{27}$$

where $a^j = a^j(x, t) = \sum_{k \geqslant 0} a^j_k(x) t^k$, and t is again one-dimensional. All considerations are local near $x = 0$, $t = 0$.

We are interested in solutions which blow up on Σ defined by $t = \psi(x)$; we seek $u \sim (t - \psi(x))^{-p/q} v_0(x)$ for integers p and q as below.

Four technical assumptions are now described. The role of our four assumptions is as follows:

(1) ensure that the blow-up surface is non-characteristic;
(2) require power growth for the nonlinearity;
(3) express that it is possible to compute the leading term so as to balance the derivatives with the nonlinearity;
(4) ensure that the resonances are constant.

Σ is required to be non-characteristic (as is usual in the WTC procedure):

$$Q(x) = \left(1 + \sum_j a_0^j \partial_j \psi\right) \quad \text{is invertible.} \tag{28}$$

We require that $b(u)$ have power growth at infinity: there are integers p and q, with $q > 0$, such that $\tau^{p+q} b(\tau^{-p}\xi)$ is analytic in $\tau \in \mathbb{C}$ and $\xi \in \mathbb{C}^m$, near $\tau = 0$, $\xi = 0$. We write

$$\tau^{p+q} b(\tau^{-p}\xi) = c(\tau, \xi) := \sum_{j \geqslant 0} c_j(\xi)\tau^j. \tag{29}$$

Substitution of the leading behaviour leads to

$$-pv_0 = q\,Q(x)^{-1} c_0(v_0) \tag{30}$$

which we assume has a non-trivial solution.

Finally, we require that there exist a matrix-valued function $P(x)$ such that

$$P^{-1} Q^{-1} c_0'(v_0) P \quad \text{is constant.} \tag{31}$$

(Here, c_0' is the matrix of derivatives of c_0 with respect to the components of u.)

Let us show that the assumptions (28)–(31) ensure that one can indeed reduce the system to a generalized Fuchsian system of the type considered in step 2.

It is convenient to introduce the new time variable $T = t - \psi(x)$, and to write the equation as

$$Qu_T = a(\partial u) + b(u) + (a_0 - a)(\partial\psi)u_T$$

where $a(\partial u) = \sum_j a^j \partial_j u$ and $a_0(\partial u) = \sum_j a_0^j \partial_j u$. Note that $(a_0 - a) = \mathrm{O}(T)$. We wrote ∂u for all the first-order spatial derivatives of u. Next, since we are in a 'weak Painlevé' situation, we let $T = \tau^q$ and $u = v\tau^{-p}$; using the assumption on $b(u)$, we find:

$$Q(\tau v_\tau - pv)/q = \tau^q a(\partial v) + c(\tau, v) + (a_0 - a)(\partial\psi)(\tau v_\tau - pv)/q.$$

Since, by (28), Q^{-1} exists, we have $(Q - (a_0 - a)(\partial\psi))^{-1} = Q^{-1} + \mathrm{O}(T) = Q^{-1} + \tau^q R$, and we find

$$\tau v_\tau - pv = q(Q^{-1} + \tau^q R)[\tau^q a(\partial v) + c(\tau, v)]. \tag{32}$$

We now substitute

$$v = v_0 + \vec{\tau} \cdot w := v_0 + \tau_0 w_0 + \cdots + \tau_l w_l$$

where $\tau_j = \tau(\ln\tau)^j$; thus, $\tau_0 = \tau$. We use $\vec{\tau}$ to denote $(\tau_0, \ldots, \tau_l)$. We find, using (29), that

$$c(\vec{\tau}, v) = c(\tau_0, v_0 + \vec{\tau} \cdot w) = c_0(v_0) + c_0'(v_0)[\vec{\tau} \cdot w] + \tau_0 c_1(v_0) + \sum_k \tau_k \vec{\tau} \cdot h_k(\vec{\tau}, x, w, \partial w).$$

It will be convenient to write $\vec{\tau} \cdot [c_0'(v_0)w]$ for $c_0'(v_0)[\vec{\tau} \cdot w]$, which amounts to defining

$$c_0'(v_0)w = (c_0'(v_0)w_0, \ldots, c_0'(v_0)w_l).$$

The calculation of $\tau\partial_\tau v = N(\vec{\tau} \cdot w)$, where $N = \sum_k (\tau_k + k\tau_{k-1})\partial/\partial\tau_k$, is identical to that of step 2, namely

$$(N - p)(\vec{\tau} \cdot w) = \sum_{j=0}^{l} \tau_j \{(N - p)w_j + w_j + (j + 1)w_{j+1}\}.$$

We also note that since $q \geqslant 1$, there exist φ_1 and h_1 such that

$$q(Q^{-1} + \tau_0^q R)(\tau_0^q a(\partial v_0)) = \vec{\tau} \cdot \left(\delta_{j0}\varphi_1(x) + \sum_k \tau_k h_{1k}(\vec{\tau}, x, w, \partial w)\right)$$

while there exist φ_2 and h_2 such that

$$q(Q^{-1} + \tau_0^q R)\left[\tau_0^q a(\vec{\tau} \cdot \partial w) + c_0(v_0) + \vec{\tau} \cdot [c_0'(v_0)w] + \sum_k \tau_k \vec{\tau} \cdot h_k\right]$$
$$= qQ^{-1}c_0(v_0) + \vec{\tau} \cdot \left\{qQ^{-1}[c_0'(v_0)w] + \delta_{j0}\varphi_2(x) + \sum_k \tau_k h_{2k}(\vec{\tau}, x, w, \partial w)\right\}.$$

We are ready to write (32), which now becomes

$$(N - p)(\vec{\tau} \cdot w) - pv_0$$
$$= q(Q^{-1} + \tau_0^q R)\Big[\tau_0^q (a(\partial v_0) + \vec{\tau} \cdot a(\partial w)) + c_0(v_0) + \vec{\tau} \cdot [c_0'(v_0)w]$$
$$+ \tau_0 c_1(v_0) + \sum_k \tau_k \vec{\tau} \cdot h_k(\vec{\tau}, x, w, \partial w)\Big]$$
$$= qQ^{-1}c_0(v_0) + \vec{\tau} \cdot \left[qQ^{-1}[c_0'(v_0)w] + (\varphi_1 + \varphi_2)\delta_{j0} + \sum_k \tau_k(h_{1k} + h_{2k} + h_k)\right].$$

Letting $\varphi = \varphi_1 + \varphi_2$ and $g = h_1 + h_2 + h$, it is now natural to consider the system

$$(N - p - qQ^{-1}c_0'(v_0))w_j + w_j + (j+1)w_{j+1} = \varphi\delta_{j0} + \sum_k \tau_k g_{kj}$$

where g_{kj} is the jth component of g_k. Letting $w_j = Pz_j$, we obtain a system of Fuchsian form. It remains to eliminate φ by introducing more variables as necessary, by a procedure analogous to the one in subsection 3.3.

3.5. Alternative argument

While this argument is quite sufficient for the expansions considered so far, we mention a second procedure, which will be useful later.

The point is that the reduction of subsection 3.4 requires only in practice the existence of a formal series solution, even if it was obtained by a procedure other than that of section 2. It sometimes succeeds even if the leading terms do not contain u_0 alone, as in the 'degenerate Cauchy–Kowalewska situation' of section 5.

Let us say that $v = \sum_{j \leqslant g} v_g$ is a solution of

$$(N + A)u = \sum t_q f_q \tag{33}$$

up to order g, if

$$(N + A)v = \sum t_q f_q[v] + \sum_{|a|=g+1} t^a \phi_a(t, x) \tag{34}$$

for some functions ϕ_a; the dependence of the nonlinearities on the derivatives of v will be suppressed in this paragraph. Note that one may further decompose the remainder as follows:

$$\sum_{|a|=g+1} t^a \phi_a(t, x) = \sum_q t_q \sum_{|c|=g} t^c \phi_{cq}.$$

We prove the following theorem.

Theorem 3. If (33) has a solution up to order g, there is a system of the form

$$((N + A')w)_a = \sum_q t_q g_{q,a}[w]$$

which generates solutions of (33) via the substitution

$$u = v + \sum_{|a|=g} t^a w_a$$

and for which the eigenvalues of A' have the form $\lambda + g$, where λ runs through the eigenvalues of A.

Proof. It suffices to compute the result of the substitution: on the one hand, we have, on the space of homogeneous polynomials of degree g, $N = g + M$ (see section 6). Furthermore, M is nilpotent. Let us write

$$Mt^a = \sum_{|b|=g} M_{ab} t^b.$$

We then have

$$(N + A) \sum_a t^a w_a = \sum_{a,b} [(N + A + g)\delta_{ab} + M_{ab}] w_a t^b.$$

Therefore

$$(N + A) \sum_a t^a w_a = \sum_a t^a ((N + A')w)_a$$

where the eigenvalues of A' are as indicated in the theorem.

As for the nonlinear terms, there are functions $h_{q,a}$ such that

$$f_q \left(x, t, v + \sum_a t^a w_a \right) = f_q[v] + \sum_a t^a h_{q,a}(x, w, \partial w).$$

Since v is a formal solution up to order g, equation (34) holds, and we find that

$$(N + A')w_a = \sum_q t_q [h_{q,a}(x, w, \partial w) + \phi_{aq}]$$

implies that u solves the desired equation. □

4. Structure of the formal series

In this section, we consider again a single equation of the form

$$Q(t\partial_t)u = \sum_{q \leqslant k_0} t(\ln t)^q G_q[t, t\ln t, \ldots, t(\ln t)^{l_0}, u, Du, tD\partial_x u, \ldots]. \tag{35}$$

Recall that $D = t\partial_t$.

4.1. Generalities

We saw in the previous section that there is an integer l such that solutions in powers of $t(\ln t)^j$, $j \leqslant l$, exist. This means that there is a series (5) with that value of l, and which solves the original equation (8).

We give here a much more precise estimate of the optimal (i.e. smallest) value of l which enters in (5). This estimate will be called l'.

As mentioned in the introduction, it seems that the structure of logarithmic WTC series can be thought of as giving a measure of how 'non-integrable' the equation under consideration is.

4.2. Inessential functions

Note first that since the variables $(t_0, \ldots, t_l)$ play only an intermediate role, it is helpful to distinguish those functions which become zero upon replacing t_j by $t(\ln t)^j$:

Definition. We say that a polynomial (or a power series) $P(t)$ is *inessential* if

$$P(t, t\ln t, \ldots, t(\ln t)^l) \equiv 0.$$

It is proved in the appendix that the space of inessential functions is invariant under N. Of course, inessential functions may involve space variables as parameters.

A basic observation is that we may replace (18) by

$$Q(N)u = \sum_q t_q(G_q + I_q) \tag{36}$$

where I_q is any inessential polynomial. We will see that an appropriate choice of I_q will enable us to considerably lower the value of l.

4.3. Role of semi-invariants

For each resonance, the corresponding term in the formal solution contains arbitrary functions of $t_0, \ldots, t_l$. These functions must satisfy, in the notation of section 6,

$$M^r u = 0$$

where r is the multiplicity of the resonance.

Now, the homogeneous polynomials which satisfy $Mu = 0$ are known as *semi-invariants* or *sources of covariants* in the invariant theory of binary forms, see the appendix for details. Except for pure powers of t_0, they are all inessential (see the appendix). They have been classified [11]. In particular, there are such polynomials which involve any given t_l if the degree is chosen large enough. Thus, there are usually different formal solutions for every choice of l. However, theorem 4 below proves that they merely differ by inessential terms if l is large enough.

4.4. Results and proofs

We now state the results.

Let us assume that the coefficients of Q, as computed in section 2, are constant, to simplify matters. (This will be the case for all our examples.) In keeping with section 2, we will say that r is a resonance if and only if $Q(r-1) = 0$, and its multiplicity is by definition the multiplicity of $r-1$ as a root of Q.

Theorem 4. Let l' be the the sum of (i) twice the multiplicity of 1 as a resonance, or l_0 if it is greater, and (ii) the maximum multiplicity of any other positive resonance. Then there are inessential polynomials $I_0, \ldots, I_{l_0}$ such that all formal formal solutions of (18) have the form $u = u(t, \ldots, t(\ln t)^{l'})$, where $u(t_0, \ldots, t_{l'})$ solves

$$Q(N)u = \sum_q t_q (G_q + I_q(t_0, \ldots, t_{l'})).$$

The number of arbitrary functions in the resulting solution equals the sum of the multiplicities of the positive integer resonances.

Specializing to the case of simple resonances, we obtain the following corollary.

Corollary 5. If all resonances are simple and greater than 1, one may take $l = l' = 1$. More precisely, there is a formal solution of (18) of the form $u = u(t, t \ln t)$, with as many arbitrary functions as there are positive resonances.

Proofs. To say that u is a solution means that

$$Qu - \sum_q t_q G_q[u]$$

is inessential, and therefore can be written $\sum_q t_q J_q(t)$.

We therefore consider the most general series solution of this equation and show that its essential part is independent of J_q. We then compute the formal solution to some high order, and introduce the I_q. The existence and convergence of the series solution then follows from subsection 3.5.

Let us substitute

$$u = \sum_g u_g$$

where u_g is a homogeneous polynomial in $(t_0, \ldots, t_l)$, of degree g, into equation (18). Note that we consider, as in subsection 3.5, the homogeneous parts of the series solution, rather than its coefficients, for convenience.

We first prove, by induction on g, that u_g is the sum of an essential and an inessential part, the former depending on $(t_0, \ldots, t_{l'})$, where l' is defined as in theorem 3. We then show that one may introduce inessential polynomials I_q into the equation, in such a way that the resulting equation will have a solution where the inessential part is identically zero.

Step 1. The u_g must be determined recursively from equations of the form

$$Q(N)u_g = \sum_q t_q \{G_q + J_q\}_{g-1}$$

where $\{\ \}_g$ indicates that one takes the homogeneous part of degree g only.

Now, on polynomials of degree g, $N = g + M$, with $M = \sum_k k t_{k-1} \partial/\partial t_k$. Therefore, writing $Q(N) = M^k R(N)$ with $R(g) \neq 0$, the recursion relation reduces to the solution of

$$M^k R(N) v = \sum_q t_q \{G_q + J_q\}_{g-1}$$

where k is the multiplicity of g as a resonance. The properties of M which we will need are proved in the appendix.

We deal in this step with the case $k = 0$. In that case, we merely need to check that the r.h.s. has the desired form, since u_g will then be uniquely determined. Indeed, N is then invertible on the space of polynomials in $(t_0, \ldots, t_{l'})$.

Since $\{G_q\}_{g-1}$ involves only $u_0,\ldots,u_{g-1}$, we may use the induction hypothesis and write $\sum_{j<g} u_j = v(t_0, \ldots, t_{l'}) + w$, where w is inessential. It follows that

$$\begin{aligned} G_q &= G_q(t_0, \ldots, t_{l_0}, v + w, \ldots) \\ &= G_q(t_0, \ldots, t_{l_0}, v, \ldots) \\ &\quad + \int_0^1 [F_u(t, v + sw, Nv, \ldots)w + F_{Du}(t, v, Nv + sNw, \ldots)Nw + \cdots] \, ds. \end{aligned}$$

Now, w, $Nw,\ldots$, and all their derivatives, are all inessential. Since inessential functions are stable by product with other functions (i.e. they form an ideal), we see that $\{G_q + J_q\}_{g-1}$ is the sum of a polynomial in $(t_0, \ldots, t_{l'})$, and an inessential polynomial.

Step 2. We now assume $k > 0$. The earlier results about the form of G_q still hold.

Using theorem 8 of the appendix, we may now assert that the general solution has the form

$$u_g = G(t_0, \ldots, t_{k+l_0}) + \text{inessential}.$$

Therefore, we need to have $l' \geqslant k + l_0$. We also see that the essential part of u_g involves k arbitrary functions of x, because case (1) of that theorem ensures that u_g is determined, modulo inessentials, up to a combination of $t_0^g, \ldots, t_0^{g-k+1} t_1^{k-1}$. Since the solutions of $M^k v = t_q J_q$ for different J_q's differ by inessential polynomials, we see that the essential part of u does not depend on J_q.

In practice, we thus see that we have to solve at each resonance an equation of the form $M^k u =$ known, and one can make use of the special form of the r.h.s. to further reduce the value of l, as we do in section 5.

Step 3. Introduction of I_q. We now fix g very large, and let v_g be essential part of the formal solution we just computed, truncated at order g.

Define I_q (of degree g) so that v_g is a formal solution up to order g of

$$Q(N)v_g = \sum_q (G_q[v_g] + I_q).$$

We may now apply theorem 3 to conclude. Note that v_g contains arbitrary functions of x corresponding to each resonance.

This completes the proof of theorem 4. □

Step 4. Proof of corollary 5. If all resonances are simple and greater than 1 (or if 1 is a simple and compatible resonance), an important simplification is that $l_0 = k_0 = 0$: no logarithms appear in the first step of the reduction. If we assume that g is a simple resonance, and that for $j < g$, $u_g = u_g(t_0, t_1)$, we see that to find u_g, we must solve an equation of the form

$$MR(N)u_g = t_0 F_g(t_0, t_1)$$

where F_g is a polynomial of degree $g-1$, and $R(N)$ is invertible on the space of such polynomials. By case 3 of theorem 7, we may find a solution which depends only on t_0 and t_1. The argument is now finished as in the general case.

Corollary 5 is therefore proved. □

Remarks. (1) If there is a single simple resonance $r > 1$, the solution is in fact given by a series in t_0 and $t_0^{r-1}t_1$ (i.e. t and $t^r \ln t$). Indeed, since 1 is not a resonance, we have $k_0 = l_0 = 0$, and we find that the formal solution $u = \sum_j u_j$ is computed by solving recursively an equation of the form

$$Q(N)u_j = t_0 R_j(t_0, t_1).$$

R_j and u_j are independent of t_1 if $j < r$.

Now N (and therefore $Q(N)$) leaves invariant the space of polynomials in t_0 and $t_0^{r-1}t_1$; $Q(N)$ is invertible on this space.

On the other hand, the r.h.s. R_j must involve t_1 linearly if $j < 2r$. Assume by induction that the u_k for $k < j$ contain only monomials of the form $t_0^b t_1^c$, where $c \leqslant [k/r]$. Then R_j is a combination of polynomials

$$u_{j_1} \cdots u_{j_q}$$

such that $j_1 + \cdots + j_q + 1 = j$. Each of the u_{j_s} contains only monomials of the form $t_0^{b_s} t_1^{c_s}$ with $c_s \leqslant [j_s/r]$. It follows that $t_0 R_j$, and therefore u_j, contains only monomials $t_0^b t_1^c$ with

$$c = \sum_s c_s \leqslant \sum_s [j_s/r] \leqslant \sum_s \frac{j_s}{r} = (j-1)/r.$$

QED

This property may fail if 1 is a resonance.

(2) We have already seen that l_0 can be taken to be twice the multiplicity of 1 as a resonance.

5. Examples

This section contains three types of illustrations of our general results.

Subsection 5.1 deals with a class of fifth-order equations containing six parameters, which leaves room for a variety of possible formal series solutions. Three equations of this type are known to be integrable, and two other sub-families have been studied by WTC analysis in the literature (case 6 of subsection 5.1 in [8], and case (VIII) of subsection 5.2 in [12]). We construct singular solutions with a prescribed singularity surface, and a variable number of logarithmic terms, for general parameter values.

Subsection 5.2 deals with special parameter values leading to a modification of these results. In particular, one of the cases when the equation degenerates into a third-order equation passes the WTC test in its original form, and does not seem to have appeared in the literature.

Subsection 5.3 is devoted to regular solutions of some equations of this class, in case the unknown multiplies the top-order derivative. These solutions, although sometimes analytic, cannot be obtained from the Cauchy–Kowalewska theorem, but can be found via the procedure of subsection 3.5.

5.1. Fifth-order equations—general case

We apply the preceding to the class of fifth-order equations in one space dimension considered in [15], where a several applications and references were given.

The equation. It reads

$$u_t + \partial_x\{\alpha u_{xxxx} + \beta u u_{xx} + \gamma u_x^2 + \mu u_{xx} + q u^2 + r u^3\} = 0.$$

We first replace x by $x-\psi(t)$, and seek solutions singular along $\{x=0\}$. It is convenient here to call the new expansion variable x instead of t, since $\{t=0\}$ is characteristic, while $\{x=0\}$ is not.

After this change of variables, the equation reads

$$u_t - \psi'(x)u_x + \partial_x\{\alpha u_{xxxx} + \beta u u_{xx} + \gamma u_x^2 + \mu u_{xx} + q u^2 + r u^3\} = 0. \tag{37}$$

We seek u in the form

$$u = x^{-2}(u_0 + x u_1 + \cdots)$$

adding logarithmic terms as necessary.

Results. We discuss the form of the singular expansion for all the cases where one branch has four non-negative integer resonances. Apart from the fifth-order KdV, the Sawada–Kotera, and the Kaup–Kuperschmidt equations, we find, in the 'general case' when no two of the quantities α, r and $3\beta+2\gamma$ vanish, 24 other cases including one which also has three positive resonances in its other branch. The results are summarized in table 1. Note that cases 3, 10 and 16–20 possess families of sech^2 traveling waves, for appropriate values of q and μ, by the results of [15].

We then discuss the degenerate cases when this condition does not hold, which leads to eight other cases, including third-order equations. The third of these cases passes the WTC test in the sense that it has a singular expansion depending on three arbitrary functions. We also discuss for these third-order equations the existence of solutions of the form

$$x(u_0 + x u_1 + \cdots).$$

They are not given by the Cauchy–Kowalewska theorem if $\mu = 0$, since u then multiplies the top-order derivative. They can nevertheless be brought into Fuchsian form.

We now turn to a systematic WTC analysis of equation (37).

Leading term and resonance equation. We consider first the case when no two of the quantities α, r and $3\beta+2\gamma$ vanish. In that case, $u \sim u_0 x^{-2}$ is the only possible singular leading behaviour. The other cases are considered in the next subsection. Substitution of the pure power series into the equation leads to the following theorem.

Theorem 6. The leading term u_0 satisfies

$$u_0(120\alpha + 2(3\beta+2\gamma)u_0 + r u_0^2) = 0. \tag{38}$$

The resonance equation is then

$$Q(r-1) := (r+1)(r-6)(r^3 - 15r^2 + (86 + \beta u_0/\alpha)r - (240 + (6\beta+4\gamma)u_0/\alpha)) = 0. \tag{39}$$

In particular, 1 is a resonance only if

$$1176\alpha r = 85\beta^2 + 108\beta\gamma + 32\gamma^2.$$

Otherwise, $u_1 = 0$.

These statements are verified by routine (albeit lengthy) calculation. -1 and 6 are resonances in all cases.

From the form of the resonance equation, it is easy to see that there are 27 cases for which there are four non-negative integer resonances. They are in correspondence with the solutions of $r_1 + r_2 + r_3 = 15$. For each set of resonances, using the equation for u_0, one determines uniquely the values of $\beta u_0/\alpha$, $\gamma u_0/\alpha$, and ru_0^2/α.

To study the second branch, it is convenient to note that one can assume $u_0 = 1$ by scaling u. We assume that this has been done. The other possible value of u_0 is then $120/r$ (except in case 26 where $r = 0$, and there is only one branch). One then computes the resonances for the branch associated with this second root. The results are given in table 1, and are discussed below in more detail. Some of the more complicated entries were computed using 'Mathematica'. We recover the three known integrable cases, and find one more equation with the maximal number of positive integer resonances.

Note that q and μ do not enter at this stage.

Logarithmic terms. We are now interested in determining l' such that the singular solutions have the form

$$u = u(x, x\ln x, \ldots, x(\ln x)^{l'}, t)$$

or rather, as in section 4,

$$u = u(t_0, \ldots, t_l) = u_e(t_0, \ldots, t_{l'}) + \text{inessential}$$

with $t_j = x(\ln x)^j$.

The general statements in section 3 apply. We however give slightly sharpened statements which take into account the particular features of the equation at hand. The results are summarized in the table, and are commented below.

The main particular features of (37), which simplify the analysis, are

(1) If 1 is a resonance, it is always compatible.

(2) If 1 is not a resonance, then $u_1 = 0$ and 3 is compatible if it is a resonance. If neither 1 nor 3 is a resonance, then $u_1 = u_3 = 0$, and 5 is compatible if it is a resonance.

(3) If neither 1, 3, nor 5 is a resonance, we have $u_1 = u_3 = u_5 = 0$.

(4) If 6 is the first resonance, it is always compatible if u_1 is constant: indeed, the compatibility condition expresses the vanishing of the the coefficient of x^{-1} in the expression obtained after substitution of the series for u into the equation. But no x^{-1} term can arise by differentiation of a pure power series. Therefore, this term must come from the expansion of u_t.

(a) Since 1 is always compatible, a simple resonance at 1 does not introduce logarithms (in other words, $k_0 = l_0 = 0$). Therefore $l = 1$ is enough for all cases when all positive resonances are simple, which refers to cases 1–6, 8, 10–13, 17–19, 22 and 23.

However, we may even take $l = 0$ if resonances are compatible. This happens in the following cases.

Table 1. Fifth-order equations. List of cases with four non-negative integer resonances in one branch. It is assumed that $u_0 = 1$ for the first branch. Cases 17, 19 and 23 are integrable by IST for special values of q and μ.

	Non-negative resonances		Coefficients			
Case	First branch	Second branch	β/α	γ/α	r/α	l'
1	0, 0, 6, 15	As first branch	−86	69	120	1
2	0, 1, 6, 14	As first branch	−72	48	120	1
3	0, 2, 6, 13	As first branch	−60	30	120	1
4	0, 3, 6, 12	As first branch	−50	15	120	1
5	0, 4, 6, 11	As first branch	−42	3	120	1
6	0, 5, 6, 10	As first branch	−36	−6	120	1
7	0, 6, 6, 9	As first branch	−32	−12	120	1
8	0, 7, 6, 8	As first branch	−30	−15	120	1
9	1, 1, 6, 13	2.07772..., 6, 13.4442...	−59	127/4	107	3
10	1, 2, 6, 12	3, 6, $6+\sqrt{46}$	−48	18	96	1
11	1, 3, 6, 11	6, 12, $(87+\sqrt{20329})/58$	−39	27/4	87	1
12	1, 4, 6, 10	5, 6, $5+\sqrt{37}$	−32	−2	80	1
13	1, 5, 6, 9	6, 10, $(25+\sqrt{1345})/10$	−27	−33/4	75	1
14	1, 6, 6, 8	6, 8, $(7+\sqrt{89})/2$	−24	−12	72	1
15	1, 6, 7, 7	6	−23	−53/4	71	2
16	2, 2, 6, 11	3.94488..., 6, 12.4677...	−38	8	76	2
17	2, 3, 6, 10	5, 6, 12	−30	0	60	1
18	2, 4, 6, 9	6, 6.26589..., 11.2807...	−24	−6	48	1
19	2, 5, 6, 8	6, 8, 10	−20	−10	40	1
20	2, 6, 6, 7	6	−18	−12	36	2
21	3, 3, 6, 9	6, 12, $(39+3\sqrt{1729})/26$	−23	−21/4	39	2
22	3, 4, 6, 8	6, 8, 12	−18	−9	24	1
23	3, 5, 6, 7	6, 10, 12	−15	−45/4	15	1
24	3, 6, 6, 6	6, 12, $(3/2)(1+\sqrt{41})$	−14	−12	12	1
25	4, 4, 6, 7	6, 9.54085..., 16.2771...	−14	−11	8	2
26	4, 5, 6, 6	N/A	−12	−12	0	2
27	5, 5, 5, 6	6, 10	−11	−49/4	−5	1

- Case 17. Compatibility at level 2 imposes the relation $q + 6\mu = 0$ between q and μ. $q = \mu = 0$ corresponds to the Sawada–Kotera equation.
- Case 19. One finds $q + 6\mu = 0$ for 2 to be compatible; no further constraint is found at level 8. This is the family of fifth-order KdV equations.
- Case 23. Examination of the compatibility conditions leads to $q+3\mu = 0$. If $q = \mu = 0$, we recover the Kaup–Kuperschmidt equation.

Note that case 22 is the only other case in which the second branch has three positive integer resonances. It nevertheless requires logarithms at level 4, because the compatibility condition sets a condition on ψ.

(b) Assume that 1 is at most a simple resonance. If all resonances other than 1 are not all simple, but are at most double, we may take $l' = 2$. This corresponds to cases 7, 14–16, 20, 21, 25, 26. However, one can sometimes give a better result.

- Case 7. The first resonance at 6 is compatible, and therefore requires only $l' = 1$. Since the other resonance is simple, $l' = 1$ is enough. The same argument applies to case 21 (and also to case 16 if the compatibility condition at level 2 holds).

- Case 14. Since both 1 and 6 are compatible, only one logarithm is required for the double resonance 6, and we may take $l' = 1$.

(c) If 1 is a double resonance (case 9), we find $l_0 = 2$ by inspection, so that, since the other resonances are simple, we take $l' = 3$.

(d) The last two cases, 24 and 27, have one triple resonance, at 6 and 5, respectively. Since both are compatible, we may take $l' = 1$, instead of 3 which would have been predicted by the general rule. Let us show this for case 27, the other one being similar.

At level 5, we need to solve, for the homogeneous part of degree 5 in the solution, an equation of the form

$$M^3 v = ct_0^5.$$

But since the resonance is compatible, we actually have $c = 0$. The solution is therefore $c_0 t_0^5 + c_1 t_0^4 t_1 + c_2 t_0^4 t_2$ modulo inessentials. The coefficients c_0, c_1 and c_2 are arbitrary functions of t. But this can also be written $c_0 t_0^5 + c_1 t_0^4 t_1 + c_2 t_0^3 t_1^2$ modulo inessentials, since $t_0 t_2 - t_1^2$ is inessential. The result follows.

5.2. *Fifth-order equations—degenerate cases*

We must now consider the case when two or more among α, r or $3\beta + 2\gamma$ may vanish. The discussion breaks down into the following cases:

$\alpha = r = 0:$	$3\beta + 2\gamma \neq 0:$	$\beta \neq 0$	(I)
		$\beta = 0$	(II)
	$3\beta + 2\gamma = 0:$	$\beta \neq 0$	(III)
		$\beta = 0$	(IV)
$\alpha = 0,\ r \neq 0:$	$3\beta + 2\gamma = 0:$	$\beta \neq 0$	(V)
		$\beta = 0$	(VI)
$\alpha \neq 0,\ r = 0:$	$3\beta + 2\gamma = 0:$	$\beta \neq 0$	(VII)
		$\beta = 0$	(VIII).

In all cases, $\nu = 0, 1, 2$ is possible, and corresponds to the solutions of the Cauchy problem (if $\alpha \neq 0$, one may take $\nu = 0, \ldots, 4$). However, in the third-order case, one may not allow $\mu + \beta u$ to vanish for $x = 0$, since the equation has the form $(\mu + \beta u)u_{xxx}$ = second-order terms. In this *degenerate Cauchy–Kowalewska* situation, there may exist solutions for which $\nu = 1$. A new resonance equation can then be computed. We investigate this case separately; but first, we give below a summary of the situation for singular solutions in cases (I)–(VIII).

(IV) and (VI) correspond to the third-order KdV and modified KdV equations, which are known to have the Painlevé property.

In case (I), leading-order analysis leads to $\nu = \beta/(\beta + \gamma)$. One must compute separately the relevant compatibility condition in the exceptional cases where this ratio is an integer.

In cases (II), (V) and (VII), there is no consistent leading singular behaviour of the form $u \sim x^\nu (u_0 + \cdots)$. This does not rule out the possibility of other, more complicated, leading behaviours.

In case (III), one finds $\nu = -2$, with resonances $-1, 0$, and 6. There is a solution of the form $x^{-2} \sum_k u_k x^k$ with u_0 and u_6 arbitrary. This equation therefore passes the WTC test in its original form for this particular singular branch.

In case (VIII), the only possible singular behaviour is $u \sim u_0 x^{-4}$, with resonances at -1, 8 and 12. One must take $l = 1$ since the resonances are not compatible. The compatibility conditions have been written out by Jeffrey and Xu [8].

There are more precise results for ODE reductions (in particular if $u = u(x)$); the second-order case is of course classical; a few partial results can be found in Bureau [4] and, for related equations, in Chazy [5].

5.3. Degenerate Cauchy problems

In case $\alpha = 0$, but $\beta \neq 0$, so that the equation degenerates into a third-order equation, we have to deal with yet one more branch of solutions, namely those which vanish for $x = \psi(t)$. They are, however, not always given by the Cauchy–Kowalewska theorem.

We develop the calculations in this case in some detail, since this is another case where the leading-order balance equation does not determine the first term. As we will see, we may nevertheless re-cast the equation in Fuchsian form.

Let us first note that by adding a constant to u, and replacing t by $t - cx$ for a suitable c, one may, as we will, assume that $\mu = 0$. The equation takes the form

$$\beta u u_{xxx} + (\beta + 2\gamma) u_x u_{xx} + u_t + \{qu^2 + ru^3\}_x - \psi' u_x = 0. \tag{40}$$

We may find solutions of the form $u_0 + x u_1 + \cdots$ if $u_0 \neq 0$. We will however be interested in solutions of the form

$$u = x(u_0 + x u_1 + x^2 u_2 + \cdots). \tag{41}$$

Substitution generates at lowest order the equation

$$u_0(2(\beta + 2\gamma)u_1 - \psi') = 0$$

and, from the coefficient of x^j, $j \geqslant 1$, a recurrence relation of the form

$$(j+1)(j+2)(\beta j + \beta + 2\gamma) u_0 u_{j+1} = F_j[u_0, \ldots, u_j].$$

Thus, we find that there is a formal solution for which u_0 is arbitrary, provided that $(\beta + 2\gamma)/\beta$ is not a positive integer. In case $3\beta + 2\gamma = 0$, however, there is a resonance at level 2 which leaves the coefficient u_2 arbitrary, provided the compatibility condition

$$u_0 \psi'' + [12 q u_0^2 + 3u_0'] \psi' = 0$$

holds.

The solution is therefore very different from the case of the Cauchy problem. Let us convert the equation to Fuchsian form nevertheless.

Let us write

$$u = x u_0(t) + x^2 v \tag{42}$$

and divide through the equation by u. Using

$$u^{-1} = (x u_0)^{-1}(1 - xv/u_0 + (xv/u_0)^2 + \cdots)$$

we find, after multiplication by x, an equation for v of the form

$$Q(x\partial_x)v - \psi' = xF(x, v, xv_x, x^2 v_{xx}, v_t)$$

with

$$Q(j) = (j+1)(j+2)(\beta j + \beta + 2\gamma).$$

We therefore find that, after subtraction of the first term of the formal series (i.e. after using substitution (42)), the result is again a Fuchsian equation to which the considerations of section 3 now apply without difficulty.

Appendix. The operator M

We prove here the properties of the operator

$$M_l = \sum_{k=0}^{l} k t_{k-1} \partial/\partial t_k$$

that were used in the text. We often drop the subscript l for convenience. We also explain the role of M in the invariant theory of binary forms.

We write $\partial_k = \partial/\partial t_k$, and define the operators $G = \sum_k t_k \partial_k$, $P = \sum_k k t_k \partial_k$ and

$$M' = \sum_{k=0}^{l} (l-k) t_{k+1} \partial/\partial t_k.$$

with the convention that $t_{-1} = t_{l+1} = 0$. Note that for any monomial t^a

$$G t^a = g(a) t^a \qquad \text{and} \qquad P t^a = p(a) t^a.$$

We call $g(a)$ and $p(a)$ respectively the *degree* and the *weight* of the monomial t^a.

Theorem 7. G commutes with P, M, and M'. In addition, $\{W = lG - 2P, M, M'\}$ satisfy $[W, M] = -2M$, $[W, M'] = 2M'$, and $[M, M'] = W$.

Remark. If we let $(H, X, Y) = (-W, -M', -M)$, we obtain the standard presentation of the Lie algebra $\mathfrak{sl}(2)$.

Proof. It suffices to check these statements on monomials.

First, we note that for any monomial u of degree g and weight p, the polynomials Mu and $M'u$ are homogeneous of the same degree, but their respective weights are $p-1$ and $p+1$.

This means that the relations $[G, M] = [G, M'] = 0$, $[P, M] = -M$ and $[P, M'] = M'$ hold on all monomials, and therefore hold quite generally. Also, we have by direct calculation $[M, M'] = (lG - 2P)$. The other commutation relations follow easily from these. □

Theorem 8. Let u be a sum of monomials of the same degree g and weight p, in the variables $(t_0, \ldots, t_l)$.

(1) If $M^k v = u$ and u is inessential, homogeneous of degree g, then v is the sum of a linear combination of of $t_0^g, \ldots, t_0^{g-k+1} t_1^{k-1}$, and an inessential polynomial. Conversely, if u is inessential, so is Mu. This applies in particular to the homogeneous elements in the kernel of M.

(2) If u is a monomial with $lg - 2p > 0$, then u is in the range of M_l. In particular, any monomial in $(t_0, \ldots, t_k)$ is in the range of M_l if $l > 2k$.

(3) Assume $u = \sum_{q \leqslant k_0} t_q u'_q(t_0, \ldots, t_{l'})$, and $k_0 \leqslant l' \leqslant l$. Then the equation $M^k v = u$ can be solved, modulo inessential polynomials, by a polynomial which depends only on $(t_0, \ldots, t_{l'})$, provided that $k + k_0 \leqslant l'$.

Proof. (1) The statement is clear if $g = 0$. Let us therefore assume $g > 0$. Let $s(t) = v(t, t\ln t, \ldots)$. We have, since $N - g = M$ on polynomials of degree g,

$$t\frac{\mathrm{d}s}{\mathrm{d}t} - gs = (Mv)(t, t\ln t, \ldots) = u(t, t\ln t, \ldots) = 0.$$

Since $s(0) = 0$, $s \equiv ct^g$, so $u - ct_0^g$ is inessential. This settles the case $k = 1$. The other cases, as well as the proof of the statement in the opposite direction, are proved similarly.

(2) The statement follows from a general property of representations of $\mathfrak{sl}(2)$: the irreducible representations contained in the present one act on a chain

$$(v_k, v_{k-2}, \dots, v_{-k})$$

of polynomials of degree g, where, for every j, v_j is an eigenvector of $lG - 2P$. M maps every v_k to a non-zero multiple of v_{k+2}, resp. 0 if $k = p$, and therefore, any v_j with $j > 0$ must lie in the range of M. But these polynomials span precisely the sum of the eigenspaces of $lG - 2P$ with positive eigenvalues, as desired.

There is a direct proof of this fact in [11].

In particular, if $u = u(t_0, \dots, t_k)$, we have at any rate $p \leqslant kg$, and $l > 2k$ is certainly sufficient.

(3) It suffices to consider monomials. Let $u(\vec{t}) = \vec{t}^a = t_0^{a_0} t_1^{a_1} \cdots$, and $u(t, t\ln t, \dots) = t^g (\ln t)^{p(a)}$. We also know that $g = a_0 + \cdots + a_{l'}$, and that there is an index $q \leqslant k_0$ such that $a_q > 0$. As usual, $p(a) = a_1 + 2a_2 + \cdots$. We want, if $v(t, t\ln t, \dots) = r(t)$,

$$\left(t\frac{\mathrm{d}}{\mathrm{d}t} - g\right)^k r = t^g (\ln t)^{p(a)}$$

so that $r = \sum_{h<k} c_h t^g (\ln t)^h + t^g R(\ln t)$, where R is a polynomial of degree $p(a) + k$, and the c_h are arbitrary. Now, one can always write any expression $t^g (\ln t)^{p(a)+k}$ in the form

$$[t(\ln t)^q]^{a_q - 1} [t(\ln t)^{k+q}] \prod_{j \neq q} [t(\ln t)^j]^{a_j}$$

using the fact that $a_q > 0$. If $k + k_0 \leqslant l'$, we therefore see that we may replace t, $t\ln t, \dots$ by $t_0, t_1, \dots$ in the above expressions, to obtain a polynomial v' in $(t_0, \dots, t_{l'})$ such that $M^k v' - u$ is inessential. This is the desired result. □

Relation to invariant theory. A binary form is an expression of the form

$$p(x, y) := \sum_{k=0}^{l} \binom{l}{k} t_k x^k y^{l-k}.$$

The group SL(2) acts on the coefficients of p in the following way: if $x = ax' + by'$, $y = cx' + dy'$, where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)$$

there is another binary form, p', in x' and y' such that $p(x, y) = p'(x', y')$. Its coefficients $(t'_0, \dots, t'_l)$ define the action of the transformation on $(t_0, \dots, t_l)$.

An invariant is a function of the coefficients $(t_0, \dots, t_l)$ which remains unchanged in this transformation; a covariant has the same property, but is allowed to have homogeneous dependence on x and y.

The usefulness of this notion is that the coefficients of the top power of x in covariants coincide with the solutions of $Mu = 0$. These coefficients are called *semi-invariants*. $Mu = 0$ and $M'u = 0$ in fact express respectively, at the infinitesimal level, the invariance of u under the subgroups

$$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}.$$

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